Optimal first arrival times in Lévy flights with resetting

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We consider the diffusive motion of a particle performing a random walk with Lévy distributed jump lengths and subject to a resetting mechanism bringing the walker to an initial position at uniformly distributed times. In the limit of an infinite number of steps and for long times, the process converges to super-diffusive motion with replenishment. We derive a formula for the mean first arrival time (MFAT) to a predefined target position reached by a meandering particle and we analyze the efficiency of the proposed searching strategy by investigating criteria for an optimal (a shortest possible) MFAT.

I. INTRODUCTION

Limited random walks, with sudden termination of a trajectory are frequently analyzed in descriptions of motion in porous media, biological tissues, composite materials, and dynamic networks and of extreme, catastrophic events like gambler's ruin, chemical reactions, and species extinction [1–5]. Quite often, however, the absorption events and disappearance of trajectories are followed by resets or restart activities of the system, e.g. the relocation of searching paths in animal foraging, the seeking for target location by repair proteins or the returning to the initial position after an unsuccessful search of the address by an individual lost in a vast city [6–8].

A random walk with restart is also known as a graph mining technique widely used in the machine learning community for page-ranking or web search models and cryptology [9–12] In this approach the frequency of visits paid to a given node can be analyzed as a random walk on a graph. It is described as an ordered sequence of visits to vertices with a source (initial) vertex probability $\vec{p}_0$. For Markov chain models of transitions between subsequent locations on a graph described by a matrix Π, reset events inject additional randomness to the walk $\vec{p}_{i+1} = (1 - c)\Pi\vec{p}_i + c\vec{s}_i$ with c being the probability of resetting per step and $\vec{s}_i$ representing an arbitrary probability vector added at resetting.

Many intriguing facets of the process in which a Brownian particle is stochastically reset to its initial position with a constant rate have been investigated by Evans and Majumdar [8]. The stationary state of such a process has been shown to be described by a non-Gaussian distribution which, due to a non-vanishing steady state current directed towards the resetting position, violates the detailed-balance condition. The temporal relaxation towards this nonequilibrium steady state has been shown to exhibit a dynamical transition [13]. Moreover, it has

been proved [8] that there exists an optimal resetting rate that minimizes the average hitting time to the target. Extensions to space depending rate, resetting to a random position with a given distribution and to a spatial distribution of the target have been also considered in Ref.[14]. Brownian diffusion in external potentials have been further analyzed in a recent study by Pal [15].

In a somewhat different context, similar random walks with stochastic resets have been analyzed by Durang et al. [16] who posed the problem of interacting particles subject to a stochastic return to the initial configuration in the coagulation-diffusion process. The particles perform random hoppings to nearest-neighbour sites such that upon the encounter of two particles, the arriving particle disappears. The stochastic reset is described then by a given set of probabilities for having some consecutive empty sites. A Markov monotonic continuous time random walk model in the presence of a drift and Poisson resetting events has been addressed in an elegant work by Montero and Villarroel [17], who derived general formulas for the survival probability and the mean exit time.

While most of the works related to random walks with resets is based on continuous and discretized version of a Wiener process, relatively few studies have been devoted to resetting accompanying generalized Wiener motion with discountinuous Lévy jumps. Lévy flights and Lévy walks [18] have been claimed to be observed in many foraging animal species [19–28], which has led to theoretical analysis showing an optimality of Lévy flights or Lévy walks in different setups [20, 29–33]. The summary of those results can be found in a recently published book [34]. The optimization of a mean first passage time (MFPT) in a discrete time model of Lévy flights with stochasting resetting has been addressed in Ref.[35], where it has been shown that the optimal parameters admit jumps (i.e. discontinuous changes) as functions of a distance to the target. Hereafter, by analyzing statistics

of first arrival times (FAT) of the continuous time version of the model, we demonstrate parameter-dependent transition between the optimal Gaussian and non-Gaussian search strategies.

In this work we concentrate on a variant of the model, in which a one-dimensional jump-like searching process with resetting events is analyzed as a renewal Markov model with Lévy jumps. We assume that a random searcher starts its motion at $x_0 = 0$ and tries to find the object located at some position x. The walker does not memorize its former locations, and the steps undertaken at any instant in time are statistically independent and drawn from a symmetric stable distribution with a stability index $\alpha \in (0, 2]$. Furthermore, at random times following the Poisson point process, the searcher decides to instantaneously reset to the initial position. We derive an expression for the transition probability density of such process, analyze the existence and character of the long-time stationary distribution and discuss optimal conditions for the mean first arrival time (MFAT).

The paper is organized as follows: Section II introduces the model and discusses the structure and stationary solutions of evolution equations for corresponding probability distribution functions. The mean first arrival time in the model is introduced in Section III and its optimization is further analyzed in Section IV which presents the most important results of this work. We summarize the paper and add conclusions in Section V.

II. TIME EVOLUTION AND TRANSITION PROBABILITY

We start with an analysis of the integral equation that governs the evolution of the probability density function for the process $\{X(t), t \geq 0\}$:

$$W(x, t|x_0, t_0)dx \equiv$$
$$\text{Prob}\{x < X(t) \leq x + dx | X(t_0) = x_0\} \quad (1)$$

In the course of time $W(x, t|x_0, t_0)$ is subject to possible reset events to $x = 0$ or jumps (Lévy flights). Resets are independent from flights and occurring in time according to Poisson statistics with an average expectation time for the occurrence of the event given by r^{-1}. Note that for the purpose of analysis, we have untied the initial and resetting positions. We denote the former as x_0 and keep the latter at the origin. The overall process is time homogeneous, i.e. $W(x, t|x_0, t_0) = W(x, t - t_0|x_0, 0) \equiv W(x, t - t_0|x_0)$, so that the propagator satisfies the equation (for the derivation and a detailed discussion see Ref.[36]):

$$W(x, t|x_0) = e^{-rt}W_0(x, t|x_0) +$$
$$+ \int_0^t d\tau e^{-r\tau} r W_0(x, \tau|0) \quad (2)$$

The first term on the RHS of the above renewal equation represents the survival of the probability mass without resetting events, whereas the second term describes the evolution after the last reset. The function $W_0(x, \tau|x_0, t_0)$ denotes the probability density function (PDF) of the process when the resetting mechanism is switched off. In this case the random walk propagator fulfills equation

$$W_0(x, t|x_0) = \delta(x - x_0) \left[1 - \int_0^t \Phi(\tau)d\tau \right]$$
$$+ \int_0^t \Phi(t - t') \int_{-\infty}^{+\infty} p(x - x')W_0(x', t'|x_0)dx'dt', \quad (3)$$

where $\Phi(t)$ is the waiting time PDF, independent of the jump-length PDF $p(x - x')$. In the Fourier-Laplace space

$$W(k, s) \equiv \mathcal{F}[\mathcal{L}[W(x, t); t \to s]; x \to k],$$

the integral Eq.(3) takes the form of

$$W_0(k, s|x_0) = \frac{1 - \Phi(s)}{s} \frac{1}{1 - \Phi(s)p(k)}, \quad (4)$$

where $\mathcal{L}[f(t)] \equiv \int_0^\infty \exp(-st)f(t)dt$. We further assume that $\Phi(t)$ has a well defined mean value, $\tau_0 = \int_0^\infty t\Phi(t)dt$, and $p(x)$ is the PDF of the Lévy stable form, so that its characteristic function reads

$$\mathcal{F}[p(x)] = \exp\left[-\sigma^\alpha |k|^\alpha\right], \quad (5)$$

with the stability index $0 < \alpha \leq 2$. The resulting process is Markovian, with the variance diverging for $\alpha < 2$ and fractional moments [37] scaling like:

$$\langle |x(t)|^q \rangle \propto (Dt)^{q/\alpha}, \quad (6)$$

where $D = \sigma^\alpha/\tau_0$. The asymptotic behavior of $W_0(k, s|x_0, 0)$ can be deduced by taking the limit $k \to 0$ and $s \to 0$ which implies:

$$\Phi(s) \approx 1 - s\tau_0 + ..., \quad p(k) \approx 1 - D|k|^\alpha. \quad (7)$$

After proper rescaling of the waiting times and jumps [37], the diffusion limit of the integral Eq.(3) is obtained in the form of a space fractional Fokker-Planck equation (FFPE)

$$\frac{\partial}{\partial t}W_0(x, t|x_0) = D\frac{\partial^\alpha}{\partial |x|^\alpha}W_0(x, t|x_0), \quad (8)$$

with $\frac{\partial^\alpha}{\partial |x|^\alpha}$ denoting the symmetric Riesz space fractional derivative which represents an integro-differential operator defined as [38, 39]:

$$\frac{\partial^\alpha}{\partial |x|^\alpha}f(x) = \frac{-1}{2\cos(\pi\alpha/2)\Gamma(2 - \alpha)} \times$$
$$\times \frac{\partial^2}{\partial x^2} \int_{-\infty}^\infty \frac{f(x')}{|x - x'|^{\alpha - 1}}dx', \quad (9)$$

which has a particularly simple form in the Fourier space

$$\mathcal{F}\left[\frac{\partial^\alpha}{\partial |x|^\alpha}f(x)\right] = -|k|^\alpha \mathcal{F}[f(x)]. \quad (10)$$

The total propagator of the process $W(x,t|x_0)$ can then be obtained from Eq.(2). In the Laplace domain this equation has the form

$$W(x,s|x_0) = W_0(x,s+r|x_0) + \frac{r}{s}W_0(x,s+r|0). \quad (11)$$

In the case of Lévy flights $W_0(k,s|x_0) = \frac{e^{ikx_0}}{D|k|^\alpha+s}$. Hence $W(k,s|x_0)$ is given by

$$W(k,s|x_0) = \frac{e^{ikx_0} + \frac{r}{s}}{D|k|^\alpha + s + r} \quad (12)$$

and obeys the differential equation

$$sW(k,s|x_0) - e^{ikx_0} =$$
$$= -D|k|^\alpha W(k,s|x_0) - rW(k,s|x_0) + \frac{r}{s}. \quad (13)$$

The inverse transformation gives the FFPE describing the evolution of the total probability distribution:

$$\frac{\partial}{\partial t}W(x,t|x_0) = D\frac{\partial^\alpha}{\partial|x|^\alpha}W(x,t|x_0) -$$
$$-rW(x,t|x_0) + r\delta(x), \quad (14)$$

with initial condition $W(x,0|x_0) = \delta(x-x_0)$. Equation (14) is analogous to the Fokker-Planck equation defining a model of diffusion with stochastic resetting [8]. The difference lies in the fact that instead of a second order spatial derivative, characteristic of normal (Gaussian) diffusion, we are dealing now with a non-local fractional derivative, which describes Lévy flights. Note that the model analyzed in this paper includes the other one as a special case, for $\alpha = 2$.

Having calculated the propagator in the Fourier-Laplace space, it is straightforward to obtain a characteristic function of the stationary distribution. For the sake of simplicity, we also introduce a length scale $\lambda^\alpha \equiv \frac{D}{r}$. By definition, the stationary PDF can be then derived from the relation

$$p_s(k;\lambda,\alpha) \equiv \lim_{s\to 0} sW(k,s|x_0) =$$
$$= rW_0(k,s=r|0) = \frac{1}{1+|\lambda k|^\alpha}. \quad (15)$$

The resulting function, Eq.(15), is known as the Linnik distribution [40, 41], which is a special case of the family of geometric stable PDFs, approximating a distribution of normalized sums of *i.i.d* random variables

$$S_N = \sum_i^N x_i, \quad (16)$$

where the number of terms N is sampled from a geometric distribution, i.e. $P(N = k) = (1-p)^{k-1}p$. Summation of that type has been used, among others, in modeling energy release of earthquakes, water discharge over a dam during a flood, or avalanche dynamics [42].

The Linnik PDF can be expressed in terms of elementary functions only for $\alpha = 2$, in which case it becomes a well-known Laplace distribution:

$$p_s(x;\lambda,2) = \frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}, \quad (17)$$

with a zero mean and a variance $\text{Var}[x^2] = 2\lambda^2$. For $\alpha = 1$ the closed-form expression for the corresponding Linnik PDF can be obtained (cf. Appendix A) in terms of special functions $\text{Si}(x) \equiv \int_0^x \frac{\sin t dt}{t}$ and $\text{Ci}(x) \equiv -\int_x^\infty \frac{\cos t dt}{t}$, and in a scaled form reads:

$$\lambda p_s(\lambda x;1,1) = \left(\frac{1}{2} - \frac{\text{Si}(x)}{\pi}\right)\sin|x| - \frac{1}{\pi}\text{Ci}(|x|)\cos x. \quad (18)$$

When passing to the analysis of the first arrival times in a subsequent Section, we note here that the result (15) has been obtained earlier in Ref.[43] for a discrete time counterpart of the resetting model.

III. THE PROBLEM OF THE FIRST ARRIVAL TIME

For the stochastic process defined by Eqs.(1,2), a question of interest is in the estimation of the waiting time before the first event of a magnitude greater than a given threshold is observed. However, as has been discussed elsewhere [44–46], the superdiffusive nature of Lévy flights strongly influences the statistics of first passage times over the threshold. In particular, due to long-range Lévy jumps occurring with an appreciable probability, the trajectory of the process may cross the threshold numerous times without actually hitting it. In consequence, the statistics of first arrival times at a predefined barrier is different from the statistics of first passages over it.

Following Refs.[1, 44], we introduce the first arrival time PDF $p_{fa}(t,x)$, which describes the distribution of times T_{fa} in terms of the integral equation for the propagator $W(x,t|0,0)$:

$$W(x,t|0,0) = \int_0^t d\tau \, p_{fa}(\tau,x)W(x,t|x,\tau). \quad (19)$$

The above formula can be easily interpreted: it simply states that the process which at time t finishes up at x, has had to get to that point for the first time at some time $\tau \in (0,t)$. After that it could move freely until at time t, it came back to the very same point. The assumption of time-homogeneity ($W(x,t|x,\tau) = W(x,t-\tau|x,0)$) explains a convolution operator on the RHS of Eq.(19). The function p_{fa} is a probability density function of its first argument. The second argument denotes that the first arrival to a position x is evaluated. For readability, we skip D, r and α in the parameter list. From now

on, we also assume that the initial and reset positions coincide.

By transforming Eq.(19) into the Laplace space a simple algebraic relation is obtained:

$$W(x, s|0) = p_{fa}(s, x) W(x, s|x). \qquad (20)$$

It is important to notice that $W(x, s|x) \neq W(0, s|0)$, as the resetting mechanism introduces space inhomogeneity. Our aim is to derive a formula for the mean first arrival time (MFAT) which can be obtained from $p_{fa}(s, x)$ as follows:

$$\langle T_{fa}(x) \rangle = -\frac{\partial}{\partial s} p_{fa}(s, x)|_{s=0} = \frac{1 - p_{fa}(s, x)}{s}|_{s=0}. \qquad (21)$$

We proceed by inserting the propagator, Eq.(11), and the algebraic relation between the propagator and $p_{fa}(s)$, Eq.(20), into the formula for MFAT, Eq.(21). After straightforward algebraic manipulations we arrive at:

$$\langle T_{fa}(x) \rangle = \frac{1}{r} \left(\frac{W_0(x, s = r|x)}{W_0(x, s = r|0)} - 1 \right) =$$
$$= \frac{1}{r} \left(\frac{p_s(0; \lambda, \alpha)}{p_s(x; \lambda, \alpha)} - 1 \right). \qquad (22)$$

Note that for simplicity we use a shortened notation $W_0(x, t) \equiv W_0(x, t|0, 0)$. Eq.(22) shows that the MFAT can be expressed either in terms of the Laplace transform of the propagator of the standard Lévy α-stable process without resetting, or in terms of the stationary PDF of the process with the resetting mechanism switched on. This result is very general, since in the derivation no particular form of $W_0(x, t|x_0)$ has been assumed.

We further focus on the special case of Lévy flights. In general, the propagator of a Lévy stable process cannot be expressed in terms of an elementary function of x. Representations in terms of the Fox functions [47] and in terms of the generalized hypergeometric functions [48] are known, but they are not useful in our case. We can, however, calculate $W_0(x, s = r|x)$ and deduce from its form the range of the stability parameter α that guarantees finiteness of the evaluated MFAT:

$$W_0(x, s|x) = \frac{1}{2\pi} \int_R dk \int_0^\infty dt e^{-st} e^{-D|k|^\alpha t} =$$
$$= \frac{\Gamma(\frac{1}{\alpha}) \Gamma(1 - \frac{1}{\alpha})}{\pi \alpha D^{\frac{1}{\alpha}} s^{1 - \frac{1}{\alpha}}} = \frac{1}{\alpha \sin \frac{\pi}{\alpha} D^{\frac{1}{\alpha}} s^{1 - \frac{1}{\alpha}}}. \qquad (23)$$

For any $x \neq 0$, the propagator $W_0(x, r|0)$ is finite, since it is an integral of an oscillating function with an amplitude decreasing to zero, and can be rewritten as an alternating series. We therefore conclude from Eq.(23) that the MFAT diverges for $\alpha \leq 1$ and remains finite for $1 < \alpha \leq 2$. That apparent finiteness of the MFAT in case of Lévy flights is rather surprising, taking into account the discontinuous character of superdiffusive trajectories and thus the possibility of overshooting (i.e. jumping over the target).

A. Asymptotic behavior

The average $\langle T_{fa}(x) \rangle$ cannot be expressed in terms of elementary functions for arbitrary α. Nevertheless, we can learn something about its behavior for large and small distances x to a target. By taking a well-known expression for the asymptotic expansion of α-stable distributions [47] and transforming it to the Laplace space, or otherwise, directly expanding:

$$\frac{1}{D|k|^\alpha + s} = \sum_{n=0}^\infty \frac{(-D|k|^\alpha)^n}{s^{n+1}} \qquad (24)$$

and transforming this back from the Fourier space term by term, we obtain an asymptotic expansion of the propagator W_0 in the Laplace space (see also [49, 50] for more formal derivations):

$$W_0(x, s|0, 0) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n+1} \sin(\frac{\pi}{2} n\alpha) \frac{D^n \Gamma(n\alpha + 1)}{s^{n+1} x^{n\alpha+1}}. \qquad (25)$$

This expression is correct for $\alpha \in (1, 2)$. For $\alpha = 2$ we don't need the asymptotic expansion since in this case we have a closed-form expression:

$$W_0(x, s|0, 0) = \frac{1}{2\sqrt{Ds}} e^{-|x|\sqrt{\frac{s}{D}}} \quad (\text{for } \alpha = 2). \qquad (26)$$

One can easily verify that Eqs.(22,23) together with Eq.(26) give the same result as the one derived in [8]. We truncate the series at the first term and so obtain the large x behavior of the MFAT:

$$\langle T_{fa}(x) \rangle \propto \begin{cases} x^{\alpha+1} & ; 1 < \alpha < 2 \\ e^x \sqrt{\frac{x}{D}} & ; \alpha = 2 \end{cases} \qquad (27)$$

We may also expand the MFAT around $x = 0$ using the known expansion of the Linnik distribution [49, 50]. This leads to:

$$\langle T_{fa}(x) \rangle \approx \frac{\alpha \sin \frac{\pi}{\alpha}}{2 \sin \frac{\pi(\alpha-1)}{2} \Gamma(\alpha)} \frac{1}{r^{\frac{1}{\alpha}} D^{1-\frac{1}{\alpha}}} x^{\alpha-1} + O(x^{2\alpha-2}). \qquad (28)$$

IV. OPTIMIZATION OF THE MFAT

Given a distance to a target x, one could be tempted to determine the optimal search kinetics of this location. We choose MFAT as an objective function, and minimize it in the space of parameters (r, α). We will denote derived parameters of the efficient strategy as $r^*(x)$, $\alpha^*(x)$, respectively and the corresponding optimal MFAT as $T^*(x)$.

A. Fair comparison

Since we want to compare Lévy flights with different stability indices α, it is important to carefully choose the

parametrization of the family of jump distributions. One commonly used is $\phi(k) = e^{-|k|^\alpha}$ which in our case means fixing $D = 1$ for every α. Alas, this choice is very arbitrary and based on simplicity of a characteristic function for symmetric stable distributions. As an alternative option, we propose here a straightforward and consistent approach based on fractional moments. Let us define a random variable ξ_α to be a position of the process without resetting at time $t = 1$ (this fixes the time unit). The p-th fractional moment may be expressed as

$$\lambda_0^p = \langle |\xi_\alpha|^p \rangle = D^{\frac{p}{\alpha}} f(\alpha, p), \qquad (29)$$

where the condition $p < \alpha$ has to be satisfied in order for the fractional moment to be finite. Function $f(\alpha, p)$ is known and reads [51]

$$f(\alpha, p) = \frac{2^{p+1}\Gamma(\frac{p+1}{2})\Gamma(-\frac{p}{\alpha})}{\alpha\sqrt{\pi}\Gamma(-\frac{p}{2})}. \qquad (30)$$

We want to keep λ_0 constant (e.g. $\lambda = 1$) so our D will depend on α and p. The most natural choice of p in our case is $p = 1$ since it does not exclude any solution (in line with findings of Section III, we refer to cases with $\alpha > 1$ assuring finiteness of MFAT) and it induces an L_1 norm that is commonly used in many applications. This choice leads to the expression:

$$D(\alpha) = \left(\frac{\pi}{2\Gamma(1 - \frac{1}{\alpha})} \right)^\alpha. \qquad (31)$$

In the following we will refer to this method of comparison, based on the choice $p = 1$, as the "fair comparison". This is in contrast to the "naive comparison" based on the simplicity of characteristic function ($D = 1$).

B. Asymptotic analysis

From the asymptotic behavior of the MFAT several conclusions may be drawn: The prefactor in Eq.(28) is bounded for $\alpha \in (1, 2]$. Consequently, for given non-zero r and D it is always possible to find x small enough, so that $\alpha = 2$ minimizes the MFAT. In other words, Brownian motion is expected to be the optimal strategy at small distances to the target. In contrast, as it can be inferred from the asymptotic behavior, Eq.(27), for large enough distances the MFAT increases with x much faster for $\alpha = 2$ than for $\alpha < 2$. In this case, the Lévy motion with $\alpha < 2$ minimizes the MFAT, thus indicating a more efficient kinetics of space exploration to detect a target.

C. Random distribution of target sites

In many natural scenarios, living organisms navigate to unpredictable or randomly distributed resources. In other words, positions of the "target" is not precisely known. How is the kinetics of random search with resetting affected by the location of targets in an unknown environment? In order to address this point, we further explore the MFAT under the constraint that the searcher knows only the mean (expected) distance to the target. Accordingly, instead of a fixed x in the evaluation of the MFAT, we use the PDF that satisfies the maximum entropy principle, i.e. a Laplace distribution $p(x)$ of target positions is assumed. The MFAT in this more general setting can be calculated by averaging over possible distances:

$$\langle T_{fa}(\lambda_t) \rangle = \int_{-\infty}^{\infty} dx \langle T_{fa}(x) \rangle p(x) =$$

$$= \frac{1}{2\lambda_t} \int_{-\infty}^{\infty} dx \langle T_{fa}(x) \rangle e^{-\frac{|x|}{\lambda_t}}. \qquad (32)$$

Even though $\langle T_{fa}(\lambda_t) \rangle$ is a different function from $\langle T_{fa}(x) \rangle$, for readability we keep the same symbol for the MFAT averaged over the distribution of targets and denote that by use of a different argument, only.

As explained in the following example, such averaging over random distances to a target leads to modification of the MFAT and becomes crucial for the optimal strategy planning. Let us assume Brownian diffusion $\alpha = 2$ with the Laplace PDF of target positions characterized by the mean distance to the target $\langle |x| \rangle = \lambda_t$. In that case the MFAT is given by the formula:

$$\langle T_{fa}(\lambda_t) \rangle = \frac{1}{r} \frac{1}{\frac{\lambda}{\lambda_t} - 1}, \qquad (33)$$

where $\lambda = \sqrt{\frac{D}{r}}$. Clearly, the MFAT is finite for $\lambda \geq \lambda_t$ and optimization of Eq.(33) yields the value of the resetting frequency $r_2^*(\lambda_t) = \frac{D}{4\lambda_t^2}$. If a searcher does not know the distribution of target locations, but was able to estimate via several measurements the mean distance to the target, $\langle |x| \rangle \approx \lambda_t$, he might be prompted to use that fixed position for further optimization of the MFAT, $\langle T_{fa}(x = \lambda_t) \rangle$. The derived optimal r^*, see Eq.(B1), when applied to the system with Laplace distributed distance-to-target, would then lead to an infinite MFAT. This apparent inconsistency demonstrates that for the proper minimization of arrival times, the actual form of distance-to-target distribution $p(x)$ is indispensable.

It can be easily shown that for heavy-tailed distance-to-target distributions, the Brownian strategy always gives an infinite MFAT. In contrast, strategies with Lévy-distributed jumps ($\alpha < 2$) may provide efficient algorithms for searching, for which the MFAT remains finite as long as the $p(x)$ distribution is characterized by a finite variance. A simple example illustrating this case is optimization of the MFAT given by Eq.(32) with the Student's t-distribution of distances to the target:

$$p(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)\lambda_t} \left(1 + \frac{x^2}{\nu\lambda_t^2} \right)^{-\frac{\nu+1}{2}}. \qquad (34)$$

In this case the integral in Eq.(32) is convergent iff condition $\alpha < \nu - 1$ holds. Numerical integration of Eq.(32) for $\nu = 2.7$ and $\nu = 4$ leads to MFAT functions displayed in Figs.5 and 6.

D. Scaling

Optimal parameters $r^*(x)$, $\alpha^*(x)$ and optimal MFAT $T^*(x)$ depend on x and D. For the sake of simplicity, from now on we fix D. It will be useful to take advantage of dimensional analysis to calculate the scaling behavior of the optimal r^* and MFAT for a given α. Let $r_\alpha^*(x)$ and $T_\alpha^*(x)$ be the optimal r and the optimal MFAT for fixed x, α and D. Up to an arbitrary multiplicative constant, the only combination of x and D that has the dimension of time is $t = \frac{x^\alpha}{D}$. This leads to the following scaling equations:

$$\begin{cases} T_\alpha^*(x) = T_\alpha^*(1)x^\alpha \\ r_\alpha^*(x) = \frac{r_\alpha^*(1)}{x^\alpha}. \end{cases} \tag{35}$$

One easily verifies that these equations hold, by calculating the derivative of the MFAT (Eq.22) with respect to r, comparing it to 0, and rewriting the corresponding equation such that it contains only a function of rx^α. Scaling equations (35) also imply similar relations to be fulfilled by $T^*(\lambda_t)$:

$$\begin{cases} T_\alpha^*(\lambda_t) = T_\alpha^*(1)\lambda_t^\alpha \\ r_\alpha^*(\lambda_t) = \frac{r_\alpha^*(1)}{\lambda_t^\alpha}. \end{cases} \tag{36}$$

The above relations are used in a numerical algorithm for optimization, as explained in details in the Appendix B.

E. Results

A comparison between analytical prediction, Eq.(22), and numerical stochastic simulations has been performed and the results are displayed in Fig.1 demonstrating a perfect agreement between both approaches. Additionally, Fig.2 presents the analytically derived MFAT functions in 2-dim (α, r) parameter space.

The MFAT diverges as $r \to 0$ and $r \to \infty$ (cf. Figs. 1,2). Accordingly, a minimum of the MFAT with respect to r can be found in the interval $[0, \infty)$ and its position depends on the stability index α characterizing underlying diffusive process.

For small x MFAT values are systematically higher for non-Gaussian diffusion ($\alpha < 2$) than for the Gaussian case and the same resetting rates. Also, as displayed in Fig.1, the MFAT has a more pronounced, deeper minimum in function of r for Lévy diffusion with heavier tails (i.e. lower α's), which suggests that the Gaussian strategy is more robust to variations of r. This is, however, no longer true for large x (cf. Fig.2). In that case MFAT values for $\alpha = 2$ are higher than that for $\alpha < 2$ and the same r, at least in the vicinity of the optimal r_α^*. Moreover, in this limit Lévy flights become more resilient to changes in r, especially in the range $r \geq r_\alpha^*$.

Results displayed in Fig.2 have been further analyzed to derive minimal values of the MFAT with respect to a pair of parameters (α, r) for different values of a distance to a target, x. Consecutive Fig.3 and Fig.4 show outcomes of the optimization procedure described in Appendix B for the cases of the immobile target located at a distance x, and the target with position described by Laplace distribution with an average distance to a target λ_t, respectively.

No qualitative difference in the derived optimal MFAT values has been found between the naive and the fair comparison. We therefore present results of the numerical optimization of $\langle T_{fa}(x) \rangle$ and $\langle T_{fa}(\lambda_t) \rangle$ for the fair comparison only.

As expected, for small x (λ_t) Gaussian diffusive motion ($\alpha^* = 2$) is the optimal searching strategy. With growing distance to a target x (or λ_t) the minimum of $\langle T_{fa} \rangle$ becomes shallower, up to some point $x^* \approx 10.8$ ($\lambda_t^* \approx 3.25$), beyond which Gaussian diffusion is not efficient anymore and the optimal stability index switches to values $\alpha^* < 2$. Corresponding values of bifurcation points x^* and λ_t^* have been obtained by means of a numerical optimization procedure and are marked in Figs.3,4 with a cross sign.

The described scenario of the continuous transition between the Gaussian and non-Gaussian optimal strategies is qualitatively similar to the one investigated in Ref.[52]. In that article yet another variant of a one-dimensional Lévy flight search strategy has been analyzed: The optimization of the random search for targets has been performed with respect to the average over inverse search times. The model has been enriched with a nonzero drift term (representative of an external bias or former experience of the searcher) and no resetting mechanism has been included. Despite these differences, their plot of the optimal α^* as a function of the initial position x (see Fig. 3, Ref.[52]) at vanishing drift strength looks very similar to ours findings in Figs. 3 and 4: There exists a finite region (of relatively small x's) in which the Brownian diffusion is the most efficient strategy and the optimal α^* is always larger than 1. It seems that these observations are generic features of the analyzed optimal first arrival times.

We have also investigated the impact of the heavy-tailed distribution of distances to the target on the efficiency of the searching. The analysis of the the optimal MFAT performed in this case is illustrated in Figs.5 and 6. Presented plots indicate that the heavy-tailed distribution of distance-to-target excludes Gaussian diffusion, $\alpha = 2$, from the set of possible optimal search strategies. Moreover, in line with the analysis of Section IV C, for Lévy flights a condition $\alpha < \nu - 1$ has to be met in order to perform a successful search with a finite MFAT.
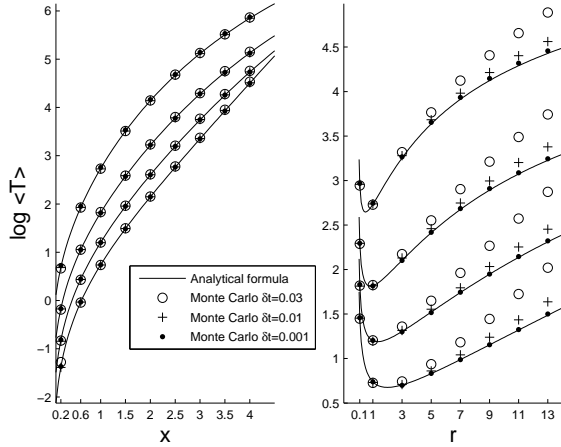
FIG. 1. Comparison between MFATs obtained by numerical integration of the analytical formula (22) (lines) and by averaging over $N = 10^5$ realizations of a simulated process. Different lines (from the top to the bottom) correspond to $\alpha = (1.4, 1.6, 1.8, 2)$. For the sake of simulation not only time has to be discretized (δt), but also a finite target size is needed. For each α the target size is chosen separately to match the analytical result at $x = 1$, $r = 1$. The same target size is further used across different values of x and r. Estimated error bars are smaller than the markers used in the plots and hence have not been displayed.
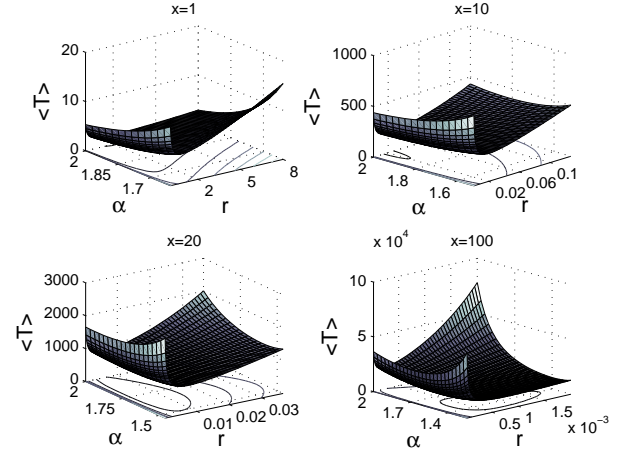

FIG. 2. The MFAT as a function of the parameters (α, r) for different values of the distance to an immobile target $x = (1, 10, 20, 100)$. Contour plots beneath the surfaces help to guide an eye towards the minimum.
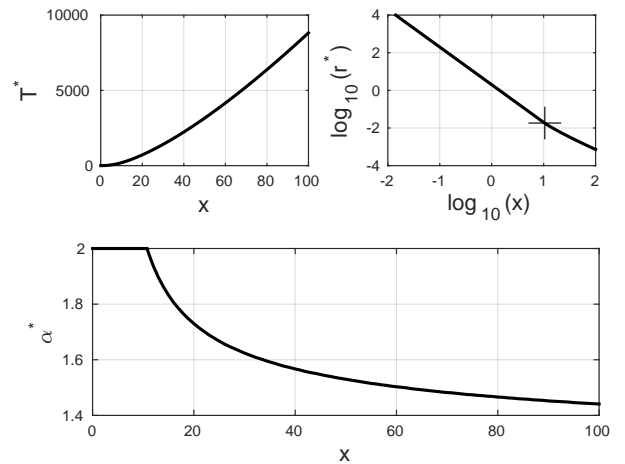

FIG. 3. Optimal parameters (α^*, r^*) and the MFAT as functions of the distance to a target x.

V. CONCLUSIONS

Not only animal foraging patterns, but also memory retrievals of humans [53] and fluctuations of their spontaneous activity [54] exhibit scaling statistics. The problem devised in this paper models mechanism of stochastic resetting, or relaxation of a diffusive searching process to a predefined threshold, and as such can be well adapted to many natural scenarios of exploration processes such as, e.g. quests for food in a given territory [55], translocation and recruitment of repair proteins seeking for a disrupted DNA strand to be repaired [56], optimal computer-aided web search [12] or statistics of recall periods in retrospective memory [53].

The efficiency of a search may be defined and analyzed by use of different measures, like e.g. the number of encounters of searchers and targets per unit of time, the mean inverse search time [52], or the exploration range of space per unit of time. Here, we have focused on the efficiency measure expressed by the mean time to reach an immobile target, the MFAT.

The first arrival time statistics has been analyzed for the one-dimensional problem with a constant resetting rate r. The acts of trajectory relocation have been assumed independent from the free (super)diffusive motion described by Lévy jumps with the exponent $0 < \alpha \leq 2$. Despite the discontinuity of trajectories, typical for Lévy flights, the MFAT remains finite iff $\alpha > 1$ with a rich

characteristics of optimal (minimal) times $T^*(x)$. By use of the designed optimization method (Section IV), we have been able to derive the optimal parameters $r^*(x)$ and $\alpha^*(x)$ for the range of target positions x. We have shown that the randomized distribution of targets with some average distance to a target results in a severe reduction of distances for which Gaussian search remains the optimal strategy. Moreover, our analysis of optimal searching times for exponential distribution of distances to a target (Section IV C) clearly indicates that not only first moment of that distribution but rather its actual form is needed for a proper optimization planning: an optimization procedure based solely on the information about the average distance to a target would result in the
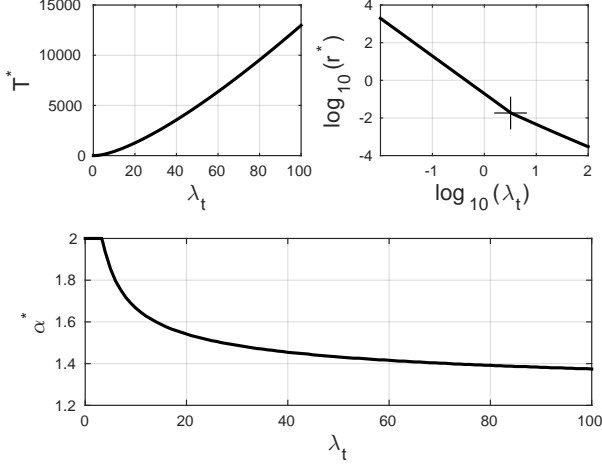
FIG. 4. Optimal parameters (α^*, r^*) and the MFAT as functions of λ_t. The target position is a random variable with a Laplace PDF of distances and an average distance-to-target λ_t.
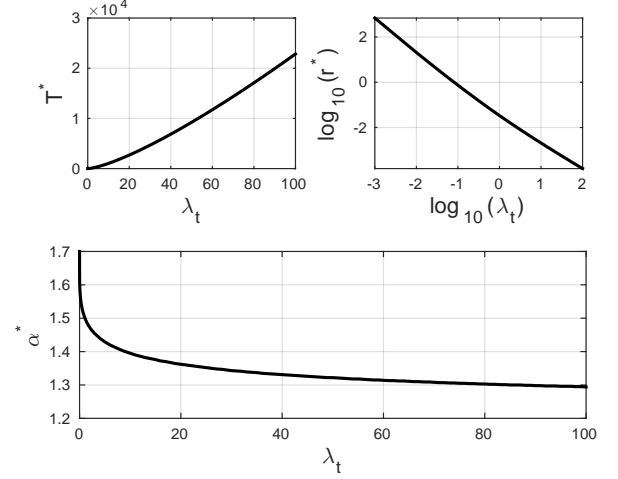

FIG. 5. Optimal parameters (α^*, r^*) and the MFAT as functions of λ_t. The target position is given by a Student's t-distribution, Eq.(34), with $\nu = 4$ and an average distance-to-target λ_t.

FIG. 6. Optimal parameters (α^*, r^*) and the MFAT as functions of λ_t. Target position is given by a Student's t-distribution, Eq.(34), with $\nu = 2.7$ and an average distance-to-target λ_t.

optimal r^* leading to an infinite MFAT.

Altogether, the proposed optimization scheme and scaling analysis can be further exploited, e.g. for two- and three-dimensional searching scenarios. Another plausible modification of the proposed procedure could be an implementation of Lévy walks, with coupled space-time distributions, or truncated Lévy flights, penalizing very long jumps.

Appendix A: Linnik distribution

The derivation of the Linnik PDF, expressed in terms of $p_s(x, \lambda, 1)$ in Eq.(18) proceeds as follows. Since $p_s(x, \lambda, 1)$ as a function of x is even, without loss of generality, we can assume that $x \geq 0$.

$$f(x) = \pi p_s(x, 1, 1) = \frac{1}{2} \int_{\mathcal{R}} dk \frac{e^{-ikx}}{1 + |k|}$$

$$= \int_0^\infty dk \int_0^\infty ds \cos(kx) e^{-s(1+k)} =$$

$$= \int_0^\infty ds \frac{se^{-s}}{x^2 + s^2} = \int_0^\infty dt \frac{te^{-tx}}{1 + t^2} =$$

$$-\frac{d}{dx} \int_0^\infty dt \frac{e^{-tx}}{1 + t^2} \equiv -\frac{d}{dx} g(x). \quad (A1)$$

One can verify that $g(x)$ is a solution of the equation

$$g''(x) + g(x) = \frac{1}{x}, \quad (A2)$$

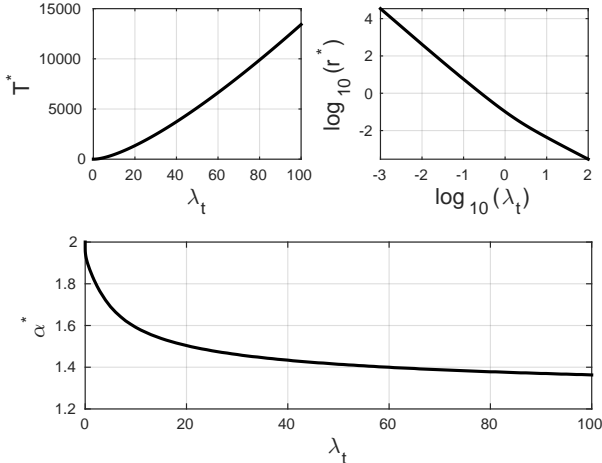which is a second order inhomogeneous linear differential equation with constant coefficients. We can easily solve

it by using the method of variation of parameters. Two constants in the general solution are calculated from the boundary conditions $g(0) = \frac{\pi}{2}$ and $\lim_{x \to \infty} g(x) = 0$. The solution reads:

$$g(x) = \left(\frac{\pi}{2} - \text{Si}(x)\right) \cos x + \text{Ci}(x) \sin x, \qquad \text{(A3)}$$

which, after differentiation, leads to formula (18).

Appendix B: Numerical scheme

The optimization problem at hand could not be solved analytically. We have thus solved it numerically. Scaling formulas Eq.(35) allow for very fast numerical optimization, by reducing numerical calculation of the MFAT to one value of x for each α and r. The algorithm then proceeds as follows: For each α we perform numerical integration by use of the reverse Fourier transform of the Linnik distribution, Eq.(15), for a given value x, e.g. $x = 1$, and a few values of r. We fit a quadratic function to the calculated points and find the minimum of that function. Next we refine the interval of r values, centering it at the estimated minimum and, consecutively, we reduce its length. This procedure is repeated until the desired accuracy is achieved. We end up with a quadratic function which, by means of its vertex coordinates, defines our $T_\alpha^*(1)$ and $r_\alpha^*(1)$. Scaling equations, Eq.(35), allow us to extend these results to arbitrary x.

When we start the calculation for a new value of α, we face the problem of choosing a proper interval of values of r. Since we fit a quadratic function, it is important that the interval contains the optimal r. For this reason, we can make use of the optimal r_{prev}^* that was calculated in a previous step, for a value of α close to the new one. Accordingly, we choose an interval of r which contains r_{prev}^*. The formula for the optimal resetting frequency for the Brownian motion case is known [8] and reads:

$$r_2^*(x) = \frac{Dz^2}{x^2}, \qquad \text{(B1)}$$

with $z \approx 1.5936$. Therefore, when performing numerical analysis, we have started our calculations from $\alpha = 2$. In the very last step we find, for each x, an α-parameter for which the smallest optimal MFAT is obtained. This is our global minimum. The numerical scheme used for the optimization of $\langle T_{fa}(\lambda_t) \rangle$ is analogous.

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